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On the Laplacian spectral radius of a tree

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Abstract

Let G be a graph; its Laplacian matrix is the difference of the diagonal matrix of its vertex degrees and its adjacency matrix. In this paper, we present a sharp upper bound for the Laplacian spectral radius of a tree in terms of the matching number and number of vertices, and deduce from that the largest few Laplacian spectral radii over the class of trees on a given number of vertices.

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1. Introduction

Let $G = (V, E)$ be a graph on vertex set $V = \{v_1, v_2, \ldots, v_n\}$. Let $|G|$ denote the number of vertices of G. The Laplacian matrix $L(G) = D(G) - A(G)$ is the difference of $D(G) = \mathrm{diag}(d(v_1), d(v_2), \ldots, d(v_n))$, the diagonal matrix of vertex degrees, and the adjacency matrix. The Laplacian spectral radius of G is the largest eigenvalue of its Laplacian matrix, denoted by $\lambda_1(G)$. Since the 1980s, many authors, for example Anderson and Morley [1], Li and Zhang [5,6], Merris [7], and Rojo et al. [9] have done a lot of work on the Laplacian spectral radius of graphs. In this paper, a sharp upper bound for the Laplacian spectral radius of a tree is given in terms of the matching number and number of vertices.

Two distinct edges in a graph G are independent if they are not incident with a common vertex in G. A set of pairwise independent edges of G is called a matching

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in G, while a matching of maximum cardinality is a maximum matching in G. The matching number $\beta(G)$ of G is the cardinality of a maximum matching of G.

Throughout this paper, we shall denote the characteristic polynomial of $L(G)$ by $\mu(G, x) = \det(xI - L(G))$.

2. Lemmas and results

Lemma 1 [2]. *The maximum eigenvalue r' of every principal submatrix (of order less than n) of a non-negative matrix A (of order n) does not exceed the maximum eigenvalue r of A. If A is irreducible, then $r' < r$ always holds. If A is reducible, then $r' = r$ holds for at least one principal submatrix.*

Lemma 2 [2]. *The increase of any element of a non-negative matrix A does not decrease the maximum eigenvalue. The maximum eigenvalue increases strictly if A is an irreducible matrix.*

Lemma 3 [3]. *Let G be a bipartite graph. Then $B(G) = D(G) + A(G)$ and $L(G)$ are unitarily similar; in particular, the maximum eigenvalue of $L(G)$ is simple provided G is connected.*

From Lemmas 1–3, we have the following result:

Corollary 1. *Let G be a connected bipartite graph, and let G' be a subgraph of G. Then $\lambda_1(G') \leqslant \lambda_1(G)$, and equality holds if and only if $G' = G$.*

Lemma 4 [3]. *Let T be a tree. Suppose v_1 and v_k are vertices each of degree at least three. Suppose the unique path $v_1, v_2, \ldots, v_k$ from v_1 to v_k is homeomorphic to an edge (i.e., apart from the endpoints, each vertex on the path has degree two). Let T' be the tree obtained from T by retracting along the entire path (i.e., deleting all edges $\{v_1v_2, v_2v_3, \ldots, v_{k-1}v_k\}$ and identifying vertices $v_1, v_2, \ldots, v_k$). Then $\lambda_1(T') > \lambda_1(T)$.*

Lemma 5 [1]. *Let G be a graph. Then $\lambda_1(G) \leqslant \max\{d(u) + d(v) : uv \in E\}$.*

Lemma 6 [8]. *If G has at least one edge, then $\lambda_1(G) \geqslant d_1(G) + 1$. For G a connected graph on $n > 1$ vertices, equality holds if and only if $d_1(G) = n - 1$.*

Let $T_m^n(2m \leqslant n + 1)$ denote the tree graph with n vertices obtained from a star $K_{1,n-m}$ by joining $m - 1$ pendants (degree 1 vertices) of $K_{1,n-m}$ to the $m - 1$ isolated vertices by $m - 1$ edges.

Theorem 1. *Let T be a tree on n vertices with matching number $\beta = \beta(T)$. Then $\lambda_1(T) \leqslant r$, where r is the maximum root of the equation*

$$x^3 - (n - \beta + 4)x^2 + (3n - 3\beta + 4)x - n = 0.$$

The equality holds if and only if $T = T_\beta^n$.

Proof. By the direct computation, we have the characteristic polynomial of the tree T_β^n is

$$\mu(T_\beta^n, x) = x(x-1)^{n-2\beta}(x^2 - 3x + 1)^{\beta-2} \times [x^3 - (n - \beta + 4)x^2 + (3n - 3\beta + 4)x - n].$$

Suppose that v_1 and v_2 are vertices of T each of degree at least three and the unique path from v_1 to v_2 is homeomorphic to an edge. Then we can apply Lemma 4 to T. Thus, a new tree T_1 can be constructed by retracting T along the entire path. Go on the above process until we obtain a tree $T_i = \tilde{T}\,(T_0 = T)$ such that we can not apply Lemma 4 to $\tilde{T}$. Then, we have

$$|\tilde{T}| \leqslant |T|, \quad \beta(\tilde{T}) \leqslant \beta(T)$$

and

$$\lambda_1(T) \leqslant \lambda_1(\tilde{T}) \quad (\lambda_1(T) = \lambda_1(\tilde{T}) \text{ if and only if } \tilde{T} = T).$$

Further, we have $\beta(T) - \beta(\tilde{T}) \leqslant |T| - |\tilde{T}|$ and there exists at most one vertex w of $\tilde{T}$ such that $d(w) \geqslant 3$. If there is no such vertex w, then $\tilde{T}$ is a path. From the above discussions, we conclude that $T = \tilde{T} = P_n$, a path with n vertices. Since $\lambda_1(P_n) = 4\sin^2((n-1)\pi/2n)$ (see [1]) and $\beta(P_n) = [n/2]$, the result is obvious. If there is such a vertex w such that $d(w) \geqslant 3$. We distinguish the following two cases:

Case 1. Assume that $\tilde{T} = K_{1,|\tilde{T}|-1}$ or $T_{\beta(\tilde{T})}^{|\tilde{T}|}$. Since $\beta(T) - \beta(\tilde{T}) \leqslant |T| - |\tilde{T}|$, then $\tilde{T}$ is an induced subgraph of T_β^n. We have from Corollary 1 that $\lambda_1(\tilde{T}) \leqslant \lambda_1(T_{\beta(T)}^n)$, the equality holds if and only if $T = \tilde{T} = K_{1,n-1}$ or $T_{\beta(T)}^n$.

Case 2. Assume that $\tilde{T} \neq K_{1,|\tilde{T}|-1}, T_{\beta(\tilde{T})}^{|\tilde{T}|}$. Then there exists a $w - v$ path of $\tilde{T}$

$$W : w = w_0, w_1, w_2, \ldots, w_{k-1}, w_k = v,$$

such that $k \geqslant 3$ and v is a pendant vertex of $\tilde{T}$.

Let $T^* = \tilde{T} - w_{k-2}w_{k-1} + ww_{k-1}$, then $|T^*| = |\tilde{T}|$. Further, we verify that $\beta(T^*) = \beta(\tilde{T})$. Suppose M is a maximum matching of $\tilde{T}$. Then either $w_{k-2}w_{k-1} \in M$ or $w_{k-1}w_k \in M$. If $w_{k-2}w_{k-1} \in M$, then $M^* = \{M - \{w_{k-2}w_{k-1}\}\} \cup \{w_{k-1}w_k\}$ is a matching of T^* and $|M^*| = |M|$; if $w_{k-2}w_{k-1} \notin M$, then $w_{k-1}w_k \in M$. Thus, M is also a matching of T^*. So, we have $\beta(T^*) \geqslant |M| = \beta(\tilde{T})$. By similar reasoning, we have $\beta(T^*) \leqslant \beta(\tilde{T})$. Hence, we have $\beta(T^*) = \beta(\tilde{T})$.

We have from Lemmas 5 and 6 that

$$\lambda_1(T^*) > d_1(T^*) + 1 = d_1(\tilde{T}) + 2 \geqslant \lambda_1(\tilde{T}).$$

Continuing the above process we have $\lambda_1(\tilde{T}) < \lambda_1\left(T^{|\tilde{T}|}_{\beta(\tilde{T})}\right)$. From the discussion of Case 1, We conclude that $\lambda_1(T) < \lambda_1\left(T^n_{\beta(T)}\right)$, and the proof is complete. $\square$

We have from Lemmas 4 and 5 that

$$\lambda_1(T^n_i) > d_1(T^n_i) + 1 = n - i + 1 = d_1(T^n_{i+1}) + 2 \geqslant \lambda_1(T^n_{i+1}). \tag{1}$$

Then, the following two results hold.

Corollary 2 [1]. *Let T be a tree of order n, then $\lambda_1(T) \leqslant n$, and equality holds if and only if $T = T^n_1 = K_{1,n-1}$.*

Corollary 3. *Let T be a tree of order n. If $T \neq K_{1,n-1}$ then $\lambda_1(T) \leqslant r_1$, where r_1 is the maximum root of the equation*

$$x^3 - (n+2)x^2 + (3n-2)x - n = 0.$$

The equality holds if and only if $T = T^n_2$.

Let $T(s,t)$ denote the tree of diameter 3 having exactly two non-pendant vertices, one of which is adjacent to s pendants and the other of which is adjacent to t pendants. In particular $n = s + t + 2$. Let

$$F = \{T(s,t) \mid s + t + 2 = n, s \geqslant 1, t \geqslant 1\}.$$

Corollary 4. *Let T be a tree with $n \geqslant 6$ vertices, $T \neq K_{1,n-1}$ and $T \neq T(n-3, 1) = T^n_2$. Then $\lambda_1(T) \leqslant r_2$, where r_2 is the maximum root of the equation*

$$x^3 - (n+2)x^2 + (4n-7)x - n = 0.$$

The equality holds if and only if $T = T(n-4, 2)$.

Proof. We have from [4] that the characteristic polynomials of $L(T(s,t))$ and $L(T(s+1, t-1))$ are

$$\mu(T(s,t), x) = x(x-1)^{n-4}[x^3 - (n+2)x^2 + (2n + st + 1)x - n],$$

$$\begin{aligned}\mu(T(s+1, t-1), x) = x(x-1)^{n-4}[&x^3 - (n+2)x^2 \\ &+ (2n + (s+1)(t-1) + 1)x - n].\end{aligned}$$

In particular,

$$\mu(T(n-4, 2), x) = x(x-1)^{n-4}[x^3 - (n+2)x^2 + (4n-7)x - n].$$

and

$$\mu(T(n-5, 3), x) = x(x-1)^{n-4}[x^3 - (n+2)x^2 + (5n-14)x - n].$$

Let

$$f_1(x) = x^3 - (n+2)x^2 + (2n + st + 1)x - n,$$

$$f_2(x) = x^3 - (n+2)x^2 + (2n + (s+1)(t-1) + 1)x - n,$$

$$f(x) = x^3 - (n+2)x^2 + (4n-7)x - n,$$

$$g(x) = x^3 - (n+2)x^2 + (5n-14)x - n.$$

If $s \geqslant t$, $x > 0$, then $f_1(x) - f_2(x) = (s - t + 1)x > 0$, we have

$$\lambda_1(T(s,t)) < \lambda_1(T(s+1, t-1)) \quad (s \geqslant t). \tag{2}$$

Let $H(s,t)$ denote trees of diameter 4 obtained from $T(s,t)$ by removing the non-pendant edge $e = uv$ and adding a new vertex w and edges uw and vw. In particular $n = s + t + 3$. Suppose $s \geqslant t \geqslant 1$. Let $H = \{H(s,t) \mid s + t + 3 = n, s \geqslant 1, t \geqslant 1\}$, then we have from Lemmas 4 and 5 that

$$\begin{aligned} \lambda_1(H(s,t)) &\leqslant d_1(H(s,t)) + 2 = s + 3 \\ &= d_1(H(s+1, t-1)) + 1 < \lambda_1(H(s+1, t-1)). \end{aligned} \tag{3}$$

By the direct computation, we have

$$\begin{aligned} \mu(H(n-4, 1), x) = x(x-1)^{n-5} \\ \times [x^4 - (n+3)x^3 + (5n-4)x^2 - (6n-10)x + n]. \end{aligned}$$

$$\mu(T_3^n, x) = x(x-1)^{n-6}(x^2 - 3x + 1)[x^3 - (n+1)x^2 + (3n-5)x - n].$$

Let

$$h(x) = x^4 - (n+3)x^3 + (5n-4)x^2 - (6n-10)x + n,$$

$$i(x) = x^3 - (n+1)x^2 + (3n-5)x - n.$$

Then

$$(x-2)i(x) - h(x) = x^2 - nx + n.$$

Note that when $h(x) = 0$, we have

$$(x^2 - 2x - 1)(x^2 - nx + n) = (x-2)[x^2 - (2n-5)x + n].$$

The right-hand side is easily seen to be negative when $4 \leqslant n - 2 \leqslant x \leqslant n - 1$ and since

$$n - 2 \leqslant \lambda_1(H(n-4, 1)) \leqslant n - 1.$$

We see that at $x = \lambda_1(H(n-4, 1))$, we have $i(x) < 0$. Since $i(n-1) > 0$, it now follows that $\lambda_1(T_3^n) > \lambda_1(H(n-4, 1))$. Since

$$i(x) - f(x) = x^2 - (n-2)x > 0, \quad (x > n-2),$$

we have $\lambda_1(T_3^n) < \lambda_1(T(n-4, 2))$. Clearly, if $\beta(T) = 2$, then either $T \in F$ or $T \in H$. From (2) and (3), we conclude that if T is a tree on n vertices with matching number two and $T \neq T(n-3, 1)$, then

$$\lambda_1(T) \leqslant \max\{\lambda_1(T(n-4, 2)), \lambda_1(H(n-4, 1))\}.$$

From (1) and Theorem 1, we conclude that if T is a tree on n vertices with matching number $\beta(T) \geqslant 3$, then

$$\lambda_1(T) \leqslant \lambda_1(T^n_{\beta(T)}) \leqslant \lambda_1(T_3^n).$$

Since

$$\lambda_1(H(n-4, 1)) < \lambda_1(T_3^n) < \lambda_1(T(n-4, 2)).$$

Then, we complete this proof. □

Corollary 5. *Let T be a tree with $n \geqslant 6$ vertices, and let $T \neq K_{1,n-1}$, $T \neq T(n-3, 1)$ and $T \neq T(n-4, 2)$. Then $\lambda_1(T) \leqslant r_3$, where r_3 is the maximum positive root of the equation $x^3 - (n+1)x^2 + (3n-5)x - n = 0$. The equality holds if and only if $T = T_3^n$.*

Proof. If $n = 6, 7$, then it is easy to show this result holds.

Suppose that $n \geqslant 8$. Since $T \neq K_{1,n-1}$, we have $\beta(T) \geqslant 2$. If $\beta(T) = 2$ and $T \neq T(n-3, 1), T(n-4, 2)$, we have from (2) and (3) that

$$\lambda_1(T) \leqslant \max\{\lambda_1(T(n-5, 3)), \lambda_1(H(n-4, 1))\}.$$

Since for $n \geqslant 8$, $x < n-1 (x \in R)$, we have $g(x) - i(x) = -x^2 + (2n-9)x > 0$. Then, for $n \geqslant 8$, $\lambda_1(T_3^n) > \lambda_1(T(n-5, 3))$. Combined with $\lambda_1(T_3^n) > \lambda_1(H(n-4, 1))$. We have $\lambda_1(T) < \lambda_1(T_3^n)$.

From (1) and Theorem 1, we conclude that if T is a tree on n vertices with matching number $\beta(T) \geqslant 3$, then

$$\lambda_1(T) \leqslant \lambda_1(T^n_{\beta(T)}) \leqslant \lambda_1(T_3^n),$$

and equality holds if and only if $T = T_3^n$. We complete this proof. □

In particular, we have

Corollary 6. *Let T be a tree on $n = 2k$ vertices with a perfect matching. Then*

$$\lambda_1(T) \leqslant \frac{k + 2 + \sqrt{k^2 + 4}}{2},$$

and equality holds if and only if $T = T_k^n$.

Proof. By the direct computation, we have the characteristic polynomial of $L(T_k^n)$ is

$$\begin{aligned}\mu(T_k^n, x) &= x(x^2 - 3x + 1)^{k-2}[x^3 - (k+4)x^2 + (3k+4)x - 2k] \\ &= x(x-2)(x^2 - 3x + 1)^{k-2}[x^2 - (k+2)x + k].\end{aligned}$$

By Theorem 1, we complete this proof. □

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References

[1] W.N. Anderson, T.D. Morley, Eigenvalues of the Laplacian of a graph, Linear and Multilinear Algebra 18 (1985) 141–145.
[2] D.M. Cvetkovic, M. Doob, H. Sachs, Spectra of graph-theoryand applications, VEB Deutscher Verlag d. Wiss. Berlin, 1979; Academic Press, New York, 1979.
[3] R. Grone, R. Merris, V.S. Sunder, The Laplacian spectrum of a graph, SIAM J. Matrix Anal. Appl. 11 (2) (1990) 218–238.
[4] R. Grone, R. Merris, Ordering trees by algebraic connectivity, Graphs Combin. 6 (1990) 229–237.
[5] J.-S. Li, X.-D. Zhang, A new upper bound for eigenvalues of the Laplacian matrix of a graph, Linear Algebra Appl. 265 (1997) 93–100.
[6] J.-S. Li, X.-D. Zhang, On the Laplacian eigenvalues of a graph, Linear Algebra Appl. 285 (1998) 305–307.
[7] R. Merris, A note on Laplacian graph eigenvalues, Linear Algebra Appl. 285 (1998) 33–35.
[8] R. Merris, Laplacian matrices of graphs: a survey, Linear Algebra Appl. 197&198 (1994) 143–176.
[9] O. Rojo, R. Soto, H. Rojo, An always non-trivial upper bound for Laplacian graph eigenvalues, Linear Algebra Appl. 312 (2000) 155–159.